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ABSTRACT

We examine the long term history of the development of fundamental representational infrastructures such as writing and algebra, and how they were physically implemented via such devices as the printing press and computers, in order to (1) gain insight into what is occurring today both in terms of representational infrastructure change and in physical embodiments, (2) obtain clues regarding what to do next, and (3) determine the kinds of questions that research will need to answer in the coming decade if we are to make optimal use of new diverse and connected classroom technologies. (Author)

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Implications of the Shift from Isolated, Expensive Technology to Connected, Inexpensive, Diverse and Ubiquitous Technologies

by

J.J. Kaput

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IMPLICATIONS OF THE SHIFT FROM ISOLATED, EXPENSIVE TECHNOLOGY TO CONNECTED, INEXPENSIVE, DIVERSE AND UBIQUITOUS TECHNOLOGIES

J. J. Kaput
University of Massachusetts-Dartmouth
<jkaput@umassd.edu>

We examine the long term history of the development of fundamental representational infrastructures such as writing and algebra, and how they were physically implemented via such devices as the printing press and computers, in order to (1) gain insight into what is occurring today both in terms of representational infrastructure change and in physical embodiments, (2) obtain clues regarding what to do next, and (3) determine the kinds of questions that research will need to answer in the coming decade if we are to make optimal use of new diverse and connected classroom technologies.

Placing current changes in historical perspective: The evolution of representational infrastructures and their material technologies

I suggest that we need to have a sense of some of the major changes of the past in order to understand the technology-related transformations of the past few decades and their trajectory through this coming decade—especially those changes having to do with the representational infrastructures with which we think and communicate. I will briefly examine the evolution of writing systems and then the printing press, as well as glance briefly towards the histories of arithmetic and algebra, in order to gain a perspective on what is happening today. In no way are the historical abstracts below intended to be definitive or at all complete, since the formal study of these matters is well beyond the scope of this paper. Further, while I will be drawing general parallels between the early development of basic representational infrastructures and the physical means by which they could be made available to wider populations, historical analogies are fraught with difficulties, especially in the details. But it is exactly in these complicating details where we can get a sense of what needs to be done to exploit the apparent representational advantages of the computational medium in mathematics education. In particular, we will, in the third part of this paper, examine some specific examples of new directions, and in the fourth part of the paper examine the kinds of open questions needing exploration in the coming few years.

Part I: The Evolution of Representational Infrastructures in Static Inert Media

The Evolution of Written Languages—Alphabetic, Phonetic Systems

Across the past half-dozen or so millennia of human history since the gradual emergence of writing—the primary means by which humanity extended its biological mind—several major changes in representational infrastructure have occurred. Over several thousand years and in several societies in the Middle East, and at various later times in other places around the world (Woodard, 1996), writing began as an ideographic, non-phonetic system for expressing ideas. In the Middle East the information was often quantitative information (Schmandt-Besserat 1978; 1981; 1988; 1992; 1994) expressed for economic purposes. These latter systems, with roots in the impressing of physical tokens in clay, made large demands on human memory and interpretive skill, and hence were laboriously learned and used only by

specialists—scribes. For example, approximately 15% of all the 100,000 existing cuneiform tablets were used to train scribes. This writing system used hundreds of non-phonetic symbols in a highly nonlinear and context-dependent way (Walker, 1987). Note also that the complex non-phonetic system and the lexical lists used to train scribes during the 3rd millennium B.C. remained essentially unchanged for more than 600 years—a hint that the conservative nature of education is not a recent development. Indeed, the time scale of evolution of writing systems is on the order of thousands of years, which suggests that the achievements did not come easily and were driven and constrained by many factors beyond the inventiveness of the scribes.

Over a period of almost 3000 years, the systems of writing in the Middle East, including Egyptian hieroglyphs, gradually evolved into more phonetic systems (although the early hieroglyphs were more pictorial than early cuneiform writing). This allowed the users to tap into the meaning-carrying and meaning-making resources of spoken language, which had evolved over the previous 300,000 years and hence had deep biological support in the muscular and neurophysiological structure of all normal humans (Deacon, 1997). However, there was never a direct map onto the sound-stream of speech, but rather a far more complex relationship that typically involved decoding the meaning of a symbol sequence prior to being able to specify the sounds associated with it (Davies, 1987). Similar evolutions occurred with many other writing systems across the world (Woodard, 1996).

So, while the move to a phonetic system improved expressiveness of the writing systems, it did not solve the problem of learnability. The solution, which appeared gradually, took the form of a small, efficient alphabet. From two to four thousand years ago, across the Middle East and Mediterranean basin, various Semitic scripts developed single-consonant syllabaries (none for vowel sounds) enabling increasingly consistent encoding of symbols for sounds that in turn carried the meanings expressed by spoken language. As happened elsewhere, these evolved over many hundreds of years in different Middle East locations to become the Arabic, Hebrew, Aramaic, and Phoenician alphabets, which provided genuinely phonetic mappings onto the sound stream—achieving a mapping of time onto space (Ong, 1982).

During the period between 700–1100 BC, the 22 consonants of the Phoenician alphabet were adopted by the Greeks, where they were extended to encode their Indo-European language by converting certain unused consonants to vowels, to take into account the sounds of their Indo European language, which gives more prominence to vowel sounds. So by reinterpreting certain Phoenician consonants as vowels, and adding three more consonants, the Greeks produced a version of the alphabet that, in various forms, we use in virtually all Western languages today. Hence, by using the sound-mapping rules of the language at hand, every written combination of these symbols can now be spoken aloud, whether or not it corresponds to the words of that language, and any idea that can be spoken in that language can be written down! As with the development of spoken language, this extraordinary achievement changed the nature of what it means to be human by changing cognition, culture and the societies in which writing ensued (Donald, 1991). As put by Goody & Watt (1968, p.9) and cited in Haas, (1996, p.11): "the notion of representing a sound by a graphic symbol is itself so stupefying a leap of the imagination that what is remarkable is not so much that it happened relatively late in human history, but that it ever happened at all" (Havelock, 1982).

Of course, there are logograms that carry meaning independent of speech sounds, including the combinations of Hindu-Arabic numerals that express numbers (whose pronunciation varies according to the phonetic language used), and various visual icons such as the circled picture of a cigarette crossed by a diagonal line that is internationally used as a "no smoking" sign. But these are at the boundary of an extraordinarily efficient symbol system that uses roughly two dozen alphabet marks with an amazing, almost infinite, range of expressiveness.

However, it was not until writing was able to tap into human speech capability *efficiently* via a highly compact alphabetic system that writing could have a chance to become universal (although the coding/decoding processes are decidedly complex and change quite radically as learning occurs, e.g., Langer, 1986). It became a fundamental representational infrastructure, learnable by most humans by the age of eight or nine, if given the opportunity. And it changed the means by which humans constructed their world individually (Nelson, 1996) and culturally (e.g., Cole, 1997; Donald, 1991). Humans became able to communicate, build, and accumulate knowledge (and all that comes with knowledge—including power and control) across time and space.

The Next Step: Making the Representational Infrastructure Widely Available via the Printing Press—An Evolutionary, Not Revolutionary, Impact

In the West, the next 1500 years saw the availability of the writing infrastructure limited by two apparently linked factors, the physical scarcity of written materials due to the lack of inexpensive reproduction technology and social conditions that limited the availability of literacy instruction primarily to elite males—elite by virtue of belonging to the ruling class or by virtue of belonging to the academic (religious) class (Kaestle, 1985; Kaufer & Carley, 1992; Resnick & Resnick, 1977). The Kaufer & Carley study shows how the widely used argument for the "revolutionary" impact of the printing press provided by Eisenstein's massive and widely accepted historical account (Eisenstein, 1979) ignores the evolutionary nature of the change that actually seems to have occurred across the three hundred years following Gutenburg. For example, for the first 100 years the traditional manuscript/scribe structure existed in parallel to the printing system, and it was not until the much faster (by 1–2 orders of magnitude) steam-driven press was available during the Industrial Revolution two hundred years later that printed materials became widely available for the kinds of documents (e.g., newspapers and political statements) that would have large social impact. Indeed, as pointed out by Haas (1996), Eisenstein's own arguments and detailed historical accounts help make the argument that the cultural, social, and political changes that accompanied the spread of printed materials—especially in vernacular languages—were gradual and related to multiple factors beyond the innovation of moveable type (which actually was used by the Chinese, along with paper, at least 500 years earlier). Furthermore, as argued in detail by Clanchy (1979), many preconditions seemed to be necessary that involved shifts in government and business needs from oral to written modes, acceptance of written records of events, distribution and communication channels, availability of paper (as opposed to expensive parchment), and other factors.

Hence, the physical technology of the printing press *did* have profound effects on society, *did* make the democratization of literacy possible, *did* lead to a standardization of language, and *did* participate in the deep cultural changes of the Renaissance leading to Modernism as outlined by Eisenstein, but its effects were intimately linked to a

variety of other changes, both prior to and following its advent in the 15th century, including changes in the printing technology itself (Kaufer & Carley, 1992).

One important distinction should be kept in mind—the distinction between a change in representational infrastructure, such as alphabetic writing, and a change in the material means by which that infrastructure can be embodied, such as the printing press and inexpensive paper—which participates in a different kind of infrastructure, a combined technological and social infrastructure.

Two Additional Representational Infrastructures—The Operative Symbol Systems of Arithmetic and Algebra

The histories of arithmetic and algebra are well-known, and we will not recount them here. See (Hoyles, Kaput, & Noss, in press) for a more detailed account. However, it is worth noting that the evolution of each was a lengthy process, covering thousands of years before the achievement of an efficient symbol system upon which a human could operate. Unlike written language, which supported the creation of fixed records in static, inert media, the placeholder system of arithmetic that stabilized in the 13th–14th centuries supported rule-based *actions* by an appropriately trained human upon the physical symbols that constitute quantitative operations on the numbers taken to be represented by those symbols. This system and the algorithms built on it, seems to be optimal in an evolutionary sense similar to the way the alphabetic phonetic writing systems seem to be optimal. Each has remained relatively stable for many centuries and has spread widely across the world. The arithmetic system, although initially a specialist's tool—for accounting purposes—came to be part of the general cultural tool-set as needs for numerical computation arose in Western societies. Interestingly, the early algorithms developed for accounting in the 14th–15th centuries and that appeared in the first arithmetic training books at that time have remained essentially unchanged to this day, and continue to dominate elementary school mathematics (Swetz, 1987).

Algebra began in the times of the Egyptians in the second millennium BC as evidenced in the famous Ahmes Papyrus by using available writing systems to express quantitative relationships, especially to "solve equations"—to determine unknown quantities based on given quantitative relationships. This is the so-called "rhetorical algebra" that continued to Diophantus' time in the 4th century of the Christian era, when the process of abbreviation of natural language statements and the introduction of special symbols began to accelerate. Algebra written in this way is normally referred to as "syncopated algebra." By today's standards, achievement to that point was primitive, with little generalization of methods across cases and little theory to support generalization.

Then, in a slow, millennium-long struggle involving the co-evolution of underlying concepts of number, algebraic symbolism gradually freed itself from written language in order to support techniques that increasingly depended on working with the symbols themselves according to systematic rules of substitution and transformation—rather than the quantitative relations for which they stood. Just as the symbolism for numbers evolved to yield support for rule-based operations on symbols taken to denote numbers, where attention and mental operations guide actions on the *notations* rather than what they are assumed to refer to, the symbolism for quantitative relations likewise developed. Bruner (1973) refers to this as an "opaque" use of the notations rather than "transparent" use, where the actions are guided by reasoning about

the entities to which the notations are assumed to refer. In effect, algebraic symbolism gradually freed itself from the (highly functional) ambiguities and general expressiveness of natural language in order that very general statements of quantitative relations could be very efficiently expressed.

However, the more important aspects of the new representational infrastructure are those that involve the rules, the syntax, for guiding operations on these expressions of generality. These emerged in the 17th century as the symbolism became more compact and standardized in the intense attempts to mathematize the natural world that reached such triumphant fruition in the "calculus" of Newton and Leibniz (more on this below). In the words of Bochner (1966, pp. 38–39):

> Not only was this algebra a characteristic of the century, but a certain feature of it, namely the "symbolization" inherent to it, became a profoundly distinguishing mark of all mathematics to follow. ... (T)his feature of algebra has become an attribute of the essence of mathematics, of its foundations, and of the nature of its abstractness on the uppermost level of the "ideation" a la Plato.

Beyond this first aspect of algebra, its role in the expression of abstraction and generalization, he also pointed out the critical new ingredient:

> ... that various types of 'equalities,' 'equivalences,' 'congruences,' 'homeomorphisms,' etc. between objects of mathematics must be discerned, and strictly adhered to. However this is not enough. In mathematics there is the second requirement that one must know how to 'operate' with mathematical objects, that is, to produce new objects out of given ones (*ibid,* p. 313).

Indeed, Mahoney (1980, p. 142) points out that this development made possible an entirely new mode of thought "characterized by the use of an operant symbolism, that is, a symbolism that not only abbreviates words but represents the workings of the combinatory operations, or, in other words, a symbolism with which one operates."

This second aspect of algebra, the syntactically guided transformation of symbols while holding in abeyance their potential interpretation, flowered in the 18th century, particularly in the hands of such masters as Euler, to generate powerful systems of understanding the world. But this operative aspect of algebra is both a source of mathematics' power, and a source of difficulty for learners. However, another learning-difficulty factor is the separation from natural language writing and hence the separation from the phonetic aspects of writing that support tapping into the many powerful narrative and acoustic memory features of natural language. Indeed, as well known via the error patterns seen in the "Student-Professors Problem", the algebraic system is in partial *conflict* with features of natural language (Clement, 1982; Kaput & Sims-Knight, 1983). For many good reasons, traditional character string-based algebra is *not* easy to learn.

Historical Analogies with Writing: Arithmetic and Algebra

Small, Elite Literacy Community.

Thus, over an extremely long period, a new special-purpose operational representational infrastructure was developed that reached beyond the symbolic operational infrastructure for arithmetic. However, in contrast with the arithmetic system, *the algebra system was built by and for a small and specialized intellectual elite* at whose hands, quite literally, it extended the power of human understanding far

beyond what was imaginable without it. Importantly, it was designed and used by specialists without regard for its learnability by the population at large. The effect of these learnability factors did not really become felt until the latter part of the 20th century when education systems around the world began to attempt to teach algebra to the general population. Prior to the middle of the 20th century, the algebra literacy community was quite small, quite analogous to the small literacy communities of specialists associated with early writing.

The Growth of Societal Need for Writing, Arithmetic and Now, Quantitative Insight.

Early writing was a response to a social need, which grew over time to include broad expressiveness requirements to encode the rules and cultures of growing urban societies, which led to the phonetic system and eventually the alphabetic system. A similar growth occurred in the case of arithmetic, which was initially the province of specially trained "reckoners"—the accountants of the 15th and 16th centuries. (A needs-argument could be given for the initial development of the arithmetic system over the previous centuries.) But the need for arithmetic skill spread across the population over the next 200–300 year, which led it to become one of the core topics of universal schooling in most countries.

By the end of the 20th century, with the growth of the knowledge economies, the need for quantitative insight spread across the population of industrialized countries in a way analogous to the way it spread for arithmetic skill earlier. This general need combined with the politically driven need to democratize *opportunity* to learn higher mathematics, typically assumed to require knowledge and skill in algebra, has produced considerable tension in many democracies, especially the United States, where access to algebra learning has come to be seen as political right (Moses, 1995). The attempts to democratize access to traditional algebra in the Unites States have not been successful despite much work and energy. Algebra is in many ways analogous to early writing in its learnability and associated literacy community.

Deep Changes in the Nature of the Representational Infrastructure—Writing and Algebra Linking Into Existing Powerful Systems.

However, just as writing gradually tapped into another, previously established human system of meaning making and communicating, and became radically reconstituted in the process as it became phonetic, algebra may likewise be on the verge of doing so. In this case, instead of the auditory-narrative system, it is the visuo-graphic system. Although it may not have been Descartes' (or Fermat's) intent, anticipated by Oresme (Clagett, 1968), he laid the base for tapping into humans' visual perceptual and cognitive capacities previously employed only by geometry. I am reminded of Joseph Lagrange's comment "As long as algebra and geometry proceeded along separate paths, their advance was slow and their applications limited. But when these sciences joined company, they drew from each other fresh vitality and thenceforward marched on a rapid pace towards perfection," cited in Kline, (1953, p. 159). From 350 years ago to our contemporary graphs of quantitative relationships, I see an analogy to the gradual transition that occurred as writing became more phonetic—the newer ways coexist with the old in various combinations as we graph algebraic functions which are defined and input into our graphing systems via character strings. And more recently, we have been able to hot-link these in the computer medium.

Will we be able to make the next transition that might make the representation and manipulation of quantitative relationships broadly learnable? In the case of writing, this required the invention of the alphabet. Below, I shall propose an analogous step for algebra.

The Development of Material Means By Which Access to Learnable Representational Infrastructures Might Be Democratized—Analogy With the Printing Press.

Anticipating the discussion below, I will suggest that, with the emergence of computers, we are involved in an extended process analogous to the evolution associated with the development of the printing press. The first stages involved expensive and hence rare central computers, what we have known as "mainframes" and "mini computers." Then came the microcomputer and networks—connectivity. More recently, we have seen the emergence of hand-helds and, even more recently, connectivity across device-types. For computation, we are heading towards the kind of ubiquitous access and full integration into life and work that was achieved by printed writing materials that eventually occurred by the time of the Industrial revolution. But, as our cursory examination of the printing press evolution suggests, the process will take time and will depend on many other changes taking place along the way.

Part II: The Birth of New Representational Infrastructures in Dynamic, Interactive Computational Media—the Case of Calculus

The Shift from Static, Inert Media to Dynamic, Interactive Media

The systems of knowledge that form the core of what was taught in schools and universities in the 20th century were built using some representational infrastructures that evolved (e.g., alphabetic and phonetic writing) and others that were somewhat more deliberately designed, mainly by and for a narrow intellectual elite (e.g., arithmetic and, to a greater extent, operative algebra). In all cases, they were instantiated in and hence subject to the constraints of the static, inert media of the previous several millennia. But, the *computational* medium is neither static nor inert, but rather, is dynamic and interactive, exploiting the great new advance of the 20th century, autonomously executable symbolic processes—that is, operations on symbol systems not requiring a human partner (Kaput & Shaffer, in press). The physical computational medium is based on a major physical innovation, the transistor, that, in turn, was the product of the prior knowledge system (Riorden & Hoddeson, 1997). The longer term development of the computational medium is reviewed in Shaffer & Kaput (1999).

Reflecting the fact that we are in the midst of a huge historical transition, I see three profound types of consequences of the development of the new medium for carrying new representational infrastructures:

Type 1: The knowledge produced in static, inert media can become knowable and learnable in new ways by changing the medium in which the traditional notation systems in which it is carried are instantiated—for example, creating hot-links among dynamically changeable graphs equations and tables in mathematics. Most traditional uses of technology in mathematics education, especially graphing calculators and computers using Computer Algebra Systems, are of Type 1.

Type 2: New representational infrastructures become possible that enable the reconstitution of previously constructed knowledge through, for example, the new types of visually editable graphs and immediate connections between functions and simulations and/or physical data of the type developed and studied in the SimCalc Project—to be described briefly below.

Type 3: The construction of new systems of knowledge employing new representational infrastructures—for example, dynamical systems modeling or multi-agent modeling of Complex Systems with emergent behavior, each of which has multiple forms of notations and relationships with phenomena. This is a shift in the nature of mathematics and science towards the use of computationally intensive iterative and visual methods that enable entirely new forms of dynamical modeling of nonlinear and complex systems previously beyond the reach of classical analytic methods—a dramatic enlargement of the MCV that will continue in the new century (Kaput & Roschelle, 1998; Stewart, 1990).

Tracing any of these complex consequences is a challenging endeavor, particularly since they overlap in substantive ways due to the fact that knowledge is co-constituted by the means through which it is represented and used—it does not exist independently of its representation (Cobb, Yackel, & McClain, 2000). Hence, we will limit our discussion to a few cases close to our recent work in the SimCalc Project involving the Mathematics of Change & Variation (MCV), of which a subset concerns the ideas underlying Calculus. Thus we will be focusing on a Type 2 change.

Calculus

While the Greeks, most notably Archimedes—whose extraordinary computational ability compensated for the weaknesses of the available representational infrastructure in supporting quantitative computation—developed certain mainly geometric ideas and techniques, the Mathematics of Change and Variation leading to what came to be called "Calculus" evolved historically beginning with the work of the Scholastics in the 1300's through attempts to mathematize change in the world (reviewed in Kaput, 1994).

The resulting body of theory and technique that emerged in the 17th and 18th centuries, cleaned up for logical hygiene in the 19th, is now institutionalized as a capstone course for secondary level students in many parts of the world, and especially in the United States. These ultimately successful mathematizing attempts were undertaken by the intellectual giants of Western civilization, who, in so doing, also developed the representational infrastructure of algebra, including extensions to infinite series and coordinate graphs, as part of the task. Their work led to profoundly powerful understandings of the different ways quantities can vary, how these differences in variation relate to the ways the quantities accumulate, and the fundamental connections between varying quantities and their accumulation. These efforts also gave rise to the eventual formalization of such basic mathematical ideas as function, series, limit, continuity, etc. (Boyer, 1959; Edwards, 1979).

Over the past two+ centuries this community's intellectual tools, methods and products—the foundations of the science and technology that we utterly depend upon—were institutionalized as the structure and core content of school and university curricula in most industrialized countries and taken as the epistemological essence of mathematics (Bochner, 1966; Mahoney, 1980) as noted above. This content has also

been taken as the subject of computerization. That is, it has been the target of Type 1 reformulation.

As already noted, the SimCalc Project has been engaged instead in a Type 2 reformulation of the core content associated with Calculus, which we review briefly before turning to newer technological issues.

Summary of SimCalc Representational Infrastructure Changes

We summarize the core web of five representational innovations employed by the SimCalc Project, all of which require a computational medium for their realization. The fifth—not discussed below in detail—is mentioned for completeness, actually falls into Type 3. In order to save space, these innovations are illustrated in later examples, which will refer to this list. Cross-platform software, Java MathWorlds for desktop computers can be viewed and downloaded at http://www.simcalc.umassd.edu and software for hand-helds can be examined and downloaded from http://www.simcalc.com.

- **Definition and direct manipulation of *graphically defined* functions, especially piecewise-defined functions**, with or without algebraic descriptions. Included is "Snap-to-Grid" control, whereby the allowed values can be constrained as needed—to integers, for example, allowing a new balance between complexity and computational tractability whereby key relationships traditionally requiring difficult prerequisites can be explored using whole number arithmetic and simple geometry. This allows sufficient variation to model interesting situations, avoid the degeneracy of constant rates of change, while postponing (but not ignoring!) the messiness and conceptual challenges of continuous change.

- **Direct connections between the above representational innovations and simulations**—especially motion simulations—to allow immediate construction and execution of a wide variety of variation phenomena, which puts phenomena at the center of the representation experience, reflecting the purposes for which traditional representations were designed initially, and enabling orders of magnitude tightening of the feedback loop between model and phenomenon.

- **Direct, hot-linked connections between graphically editable functions and their derivatives or integrals**. Traditionally, connections between descriptions of rates of change (e.g., velocities) and accumulations (positions) are usually mediated through the algebraic symbol system as sequential procedures employing derivative and integral formulas—but need not be. In this way, the fundamental idea, expressed in the Fundamental Theorem of Calculus, is built into the representational infrastructure from the start, in a way analogous to how, for example, the hierarchical structure of the number system is built into the placeholder system.

- **Importing physical motion-data via MBL/CBL and re-enacting it in simulations, and exporting function-generated data to drive physical phenomena** LBM (Line Becomes Motion), which involves driving physical phenomena, including cars on tracks, using functions defined via the above methods as well as algebraically. Hence there is a two-way connection between phenomena and mathematical notations.

- **Use of hybrid physical/cybernetic devices embodying dynamical systems,** whose inner workings are visible and open to examination and control with rich feedback, and whose quantitative behavior is symbolized with real-time graphs generated on a computer screen.

The result of using this array of functionality, particularly in combination and over an extended period of time, is a qualitative transformation in the mathematical experience of change and variation. However, short term, in less than a minute, using either rate or totals descriptions of the quantities involved, or even a mix of them, a student as early as 6^{th}–8^{th} grade can construct and examine a variety of interesting change phenomena that relate to direct experience of daily phenomena. And in more extended investigations, newly intimate connections among physical, linguistic, kinesthetic, cognitive, and symbolic experience become possible.

Importantly, taken together, these are not merely a series of software functionalities and curriculum activities, but amount to a reconstitution of the key ideas. Hence we are not merely treating the underlying ideas of Calculus in a new way, treating them as the focus of school mathematics beginning in the early grades and rooting them in children's everyday experience, especially their kinesthetic experience, but *we are reformulating them in an epistemic way.* We continue to address such familiar fundamentals as variable rates of changing quantities, the accumulation of those quantities, the connections between rates and accumulations, and approximations, but they are experienced in profoundly different ways, and are related to each other in new ways.

These approaches are not intended to eliminate the need for eventual use of formal notations for some students, and perhaps some formal notations for all students. Rather, they are intended to provide a substantial mathematical experience for the 90% of students in the U.S. who do not have access to the Mathematics of Change & Variation (MCV), including the ideas underlying Calculus, and provide a conceptual foundation for the 5–10% of the population who need to learn more formal Calculus. Finally, these strategies are intended to lead into the mathematics of dynamical systems and its use in modeling nonlinear phenomena of the sort that is growing dramatically in importance in our new century (Cohen & Stewart, 1994; Hall, 1992; Kaput & Roschelle, 1998; Stewart, 1990).

In terms of our historical perspective, we see this current work as part of a large transition towards a much more broadly learnable mathematics of quantitative reasoning, where both the representational infrastructure is changing as well as the material means by which those more learnable infrastructures can be made widely available. Taken together, it may be that the kinds of representational innovations outlined above and illustrated in the next part of the paper constitute the development of a new "alphabet" for quantitative mathematics which might do for mathematical representation what the phonetic alphabet did for writing, particularly if coupled to the right kinds of physical implementations, which are examined below.

Part III: Affordances & Constraints of Diverse, Connected, and Ubiquitous Computing Devices

Revisiting the Analogy with Changes Made Possible by the Printing Press

As we saw above, it is one thing to have a powerful and learnable representational infrastructure, such as alphabetic writing, and it is entirely another matter to have broad access to that infrastructure. We traced briefly the gradual three-century impact of the printing press on the democratization of literacy. It is a major hypothesis of this paper that a similar change is now underway in the 21st century relative to the new representational infrastructures made possible by the computational medium. We have traced three types of changes and introduced an example of one of them targeting the Math of Change & Variation (MCV) as developed in the SimCalc Project. Others could be put forward as well—for example, Dynamic Geometry™ or dynamically manipulable data management and analysis systems for statistics.

However, it is one thing to instantiate representational infrastructure innovations on expensive and hence scarce computational devices. It is entirely another to render them materially available on inexpensive, ubiquitous devices well-integrated into the flow of life, work, and education. A message from the history of the printing press is that the change needed to democratize access to the new infrastructures will be slow and will complexly involve many aspects of our culture. Of course, "slow" in today's terms is relative to rates of change that are at least an order of magnitude faster than in previous centuries, so that 300 years may shrink to 30 or less—but not likely to 3. The integration of automobiles, telephones, and television each took about a generation to reach wide penetration in industrialized societies. Penetration of the world wide web into everyday life and intellectual work seems to have taken only about half that time.

In today's school climate, full-sized, desktop computers are a relatively costly and rare commodity, compared to, say, pencils or notepads. Therefore, most schools that have computers share their availability across many different uses and populations. Furthermore, despite falling costs for a given level of power and functionality, their maintenance cost, especially in network configurations, tends to be prohibitive for most schools to deploy them on a wide-scale basis except in the wealthiest communities. The school mathematics alternative has been the graphing calculator, which has been designed primarily to support Type 1 changes. It has typically taken the form of a full, open toolkit, isolated technologically from other computational devices, and independent of any particular curriculum, which has been supplied offline. This condition is now changing.

Illustrations of Representational Infrastructure Innovations across Multiple But Non-Networked Hardware & Software Platforms

We will now explore some of the changes that are occurring in the nature and configurations of the technologies that can support the new representational infrastructures that computational media make possible. In particular, we see a rapid growth in the availability of flexible, relatively inexpensive, and wirelessly networkable hand-held devices that can run independently produced curriculum-targeted software and hence support new kinds of teaching and learning opportunities. We will first provide a series of activity-examples that simultaneously illustrate certain of the representational innovations identified above, how they map onto radically different

hardware systems, and second, how they can be extended to operate in a networked classroom.

Determining Mean Values—SimCalc Representational Infrastructure Innovations 1–3 Above

Figure 1 shows the velocity graphs of two functions, respectively controlling one of the two elevators on the left of the figure (graphs on the desktop software are color-coded to match the elevator that they control). The downward-stepping, but positive, velocity function, which controls the left elevator, typically leads to a conflict with expectations, because most students associate it with a downward motion. However, by constructing it and observing the associated motion (often with many deliberate repetitions and variations), the conflicts lead to new and deeper understandings of both graphs and motion. The second, flat, constant-velocity function in Figure 1 that controls the elevator on the right provides constant velocity. It is shown in the midst of being adjusted to satisfy the constraint of "getting to the same floor at exactly the same time." This amounts to constructing the average velocity of the left-hand elevator which has the (step-wise) variable velocity. This in turn reduces to finding a constant velocity segment with the same area under it as does the staircase graph. In this case the total area is 15 and the number of seconds of the "trip" is 5, so the mean value is a whole number, namely, 3. We have "snap-to-grid" turned in this case so that, as dragging occurs, the pointer jumps from point to point in the discrete coordinate system. Note that if we had provided 6 steps for the left elevator instead of 5, the constraint of getting to the same floor at exactly the same time (from the same starting-floor) could not be satisfied with a whole number constant velocity, hence could not be reached with "snap-to-grid" turned on.

The standard Mean Value Theorem, of course, asserts that if a function is continuous over an interval, then its mean value will exist and will intersect that function in that interval. But, of course, the step-wise varying function is *not* continuous, and so the Mean value Theorem conclusion would fail—as it would if 6 steps were used. However, if we had used imported data from a student's physical motion, as in Figure 3, then her velocity would necessarily equal her average velocity at one or more times in the interval. We have developed activities involving a second student walking in parallel whose responsibility is to walk at an estimated average speed of her partner. Then the differences between same-velocity and same-position begin to become apparent. Additional activities involve the two students in importing their motion data into the computer (or calculator) *serially* (discussed below) and replaying them *simultaneously*, where the velocity-position distinction becomes even more apparent due to the availability of the respective velocity and position graphs alongside the cybernetically replayed motion.

Note how the dual perspectives of the velocity and position functions, both illustrated in Figure 1, show two different views of the average value situation. In the left-hand graph, we see the connection as a matter of equal areas under respective velocity graphs. In the right-hand graph, we see it through position graphs as a matter of getting to the same place at the same time, one with variable velocity and the other with constant velocity. Depending on the activity, of course, one or the other of the graphs might not be viewable or, if viewable, not editable. For example, another version of this activity involves giving the step-wise varying position function on the right and asking

the student to construct its velocity-function mean value on the left. This makes the slope the key issue. By reversing the given and requested function types area becomes the key issue. Importantly, by building in the connections between rate (velocity) and totals (position) quantities throughout, the underlying idea of the Fundamental Theorem of Calculus is always at hand.

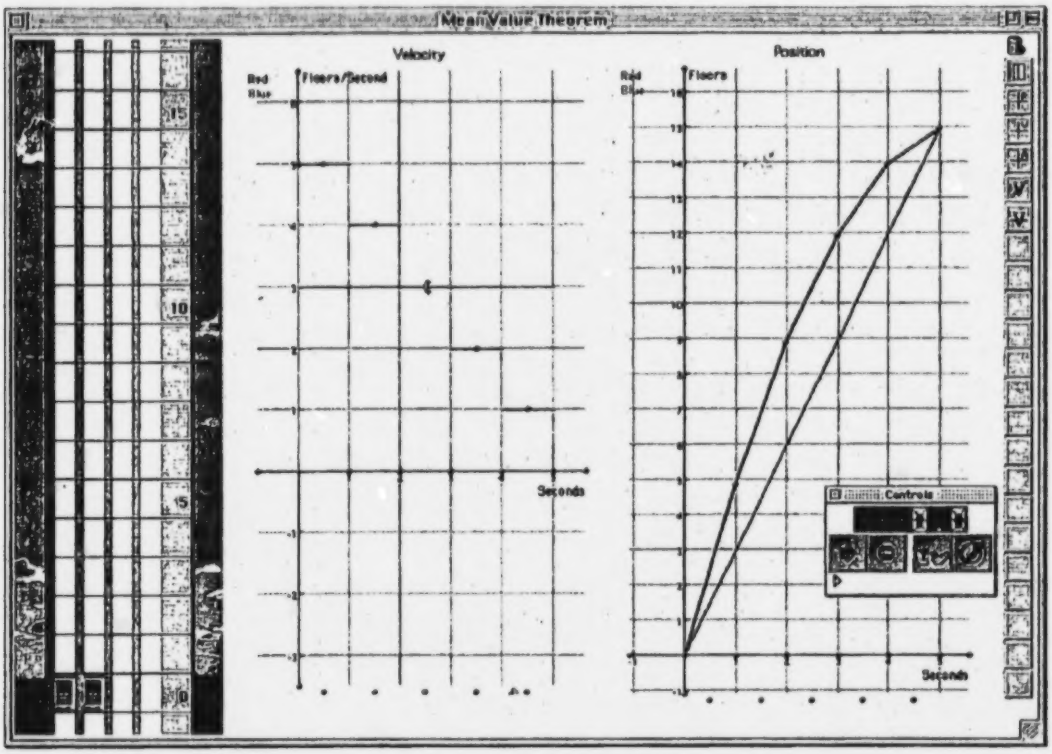

Figure 1. Averages from Both Velocity and Position Perspectives

Parallel Software and Curricula for a Graphing Calculator

Now, in the left two pictures of Figure 2 below are partially analogous software configurations—two elevators controlled by two velocity graphs. Instead of the clicking and drag/drop interface of the desktop software, most user interaction is through the SoftKeys that appear across the bottom of the screen which are controlled by the H- dKeys immediately beneath them. The left-most screen depicts the Animation Mode, with two elevators on the left controlled respectively by the staircase and constant velocity functions to their right. The middle screen depicts the Function-Edit Mode, which shows a "HotSpot" on the constant-velocity graph. The user adjusts the height and extent of a graph segment via the four calculator cursor keys (not shown), and can add or delete segments via the SoftKeys. Other features allow the user to scale the graph and animation views, display labels, enter functions in text-input mode, generate time-position output data, and so on—very much in parallel with Java MathWorlds, but without the benefits of a direct-manipulation interface. The right-most screen shows a horizontal motion world with both position and velocity functions displayed (hot-linkable if needed, as with the computer software).

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We have developed a full, document-oriented Flash ROM software system for the TI-83+ and a core set of activities embodying a common set of curriculum materials that parallels the computer software to the extent possible given the processing and screen constraints (96 by 64 pixels!). The parallelism is evident in the Calculator MathWorlds screens shown. We have also developed a prototype version of MathWorlds for the PalmPilot Operating System (see http://www.simcalc.umassd.edu to download it to a Palm device).

Figure 2. Calculator MathWorlds

Classroom/Homework Flexibility and Cognitive Flexibility

We are currently completing an Algebra and Pre-Calculus course for academically weak first year college students in which we are using a common set of activities that run on parallel versions of software for desktop computers and TI-83+ graphing calculators. The course takes place in a classroom with two computers per student and an overhead display panel for each kind of device. And each student also has a graphing calculator. Much of the classroom discussion uses the computers although some switching takes place—simply by exchanging the panel that sits on top of the overhead projector. Homework is usually assigned for the calculators, although frequently, the homework is an extension of what was begun in class on the computers. That is, the students might do the first few problems on the computer in class and then do the remaining ones with the graphing calculator at home. We imagine that there are likely to many situations where computers are only occasionally available, or where only the teacher has one, but where the students have access to hand-held devices. Hence parallel software *and* curricula can substantially expand the usability of curriculum-oriented technology in the classroom (as opposed to open tools, which likewise are useful and continue to be available in these dual-device scenarios).

Importantly, we have seen full student flexibility in switching between the two device types, despite the radically different interfaces. Careful analyses of students' discourse and gestural activity reveals that, when discussing problem solutions and difficulties, the language is primarily about the mathematical objects and relations

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rather than about the interface or device. Hence they refer to "the velocity graph" or "I need to increase the slope" or "I need to extend the domain" rather than, dragging a HotSpot, or pushing a certain key, etc. We have a suspicion, not justified at this point, that the crossing between interfaces may help in exposing the mathematical structure. After all, when a student only sees one device and one interface for working on a mathematical domain, we have every psychological reason to expect that, without reason to do otherwise, they will link their experience of that mathematics with the interface through which they learned it.

We have also found subtle perceptual carryovers from the computer to the calculator environments that may provide guidance on how to exploit the visual detail possible on the computer screen to compensate for limited screens of hand-helds. For example, despite the hard to read grid of the calculator screen, the students, who were sometimes presented activities using graph printouts based on the computer screens, seemed to treat the calculator screen as having visual attributes that were present only on the computer software. These kinds of potentially important phenomena need to be studied and documented in more detail, as do potential interference effects across the different environments.

Illustrations of Representational Infrastructure Innovations Across Multiple NETWORKED Hardware and Software Platforms

Simple Pedagogical Supports—Doing Old Stuff Better. Increasingly rich interactions are possible as connectivity increases between a teacher's computer or hand-held device, and a classroom of hand-helds. For example, a teacher can download sets of "documents" for homework or quizzes, and more interestingly, the students can upload their solution-documents as well as other data, which can then be aggregated in a variety of ways on the teacher's computer. With easy data-flow, teachers can ask diagnostic questions to 10 groups of 3 students each, such as the following (imagine the right-most part of Figure 2 above with its graphs hidden):

> The top car starts at 0 meters/second and accelerates to 3 meters/second in 6 seconds. Send me a position function for the bottom car so that it's motion matches the top car's motion exactly.

The teacher then shows all ten graphs on the same axes and then runs them. Each group's function is alive on the screen, so the diagnostic question illuminates everyone's understanding. Furthermore, patterns become evident, e.g., several groups might create a constant slope position function. Then, of course, there's a natural follow-up question—can you get your group's car to the same endpoint at exactly the same time, but at constant velocity?

New Learning Opportunities—Doing Better Stuff By Investing Individuals in a Collective Object. For example, groups of students can act out or choreograph a collective motion, say a dance, collectively, and then sit down to plan the coordination of their individual motions as mathematical functions that they will produce on their hand-held. They then upload their individual synthetically defined functions to the teacher's computer where the independently produced motions are aggregated into a simultaneously executed dance to be viewed by the entire class as in Figure 3. This kind of activity can quite literally pull students towards a parameter-based description of their motions because the motions differ in a quantitatively systematic way.

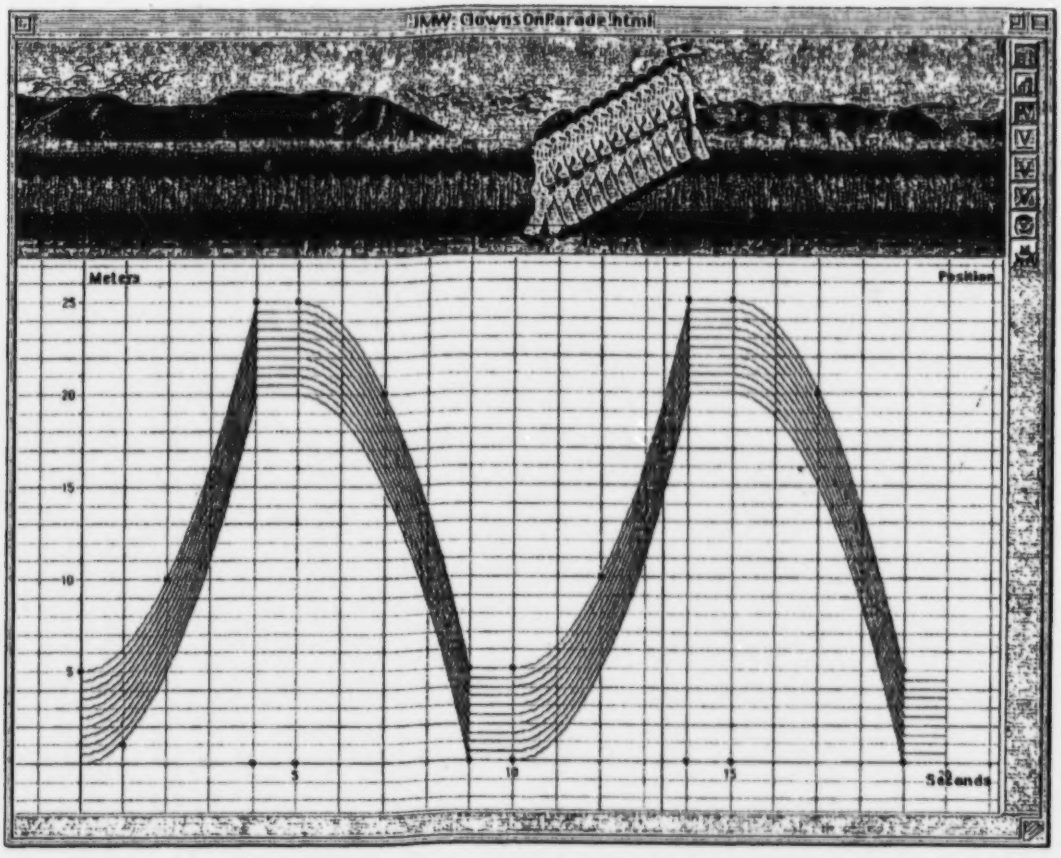

Figure 3. A Clown Parade—Staggered Initial Positions

Variations of this kind of aggregation activity can use CBL motion data input as well, where one character has its motion based on a student's actual physical motion imported into MathWorlds and where the other students create motions to participate in the parade. A wide variety of other aggregation and target activities is possible, for example, where each character's motion is imported from a serially produced dance. The kind of planning required for this kind of activity is exactly the kind of thinking that one wants in defining functions describing change. Another example involves building a "wave" action via delayed starting times as in Figure 4 where the dots hit the far "wall" at 18 meters and reflect backward. How could you make the same reflected wave motion with staggered starting positions?

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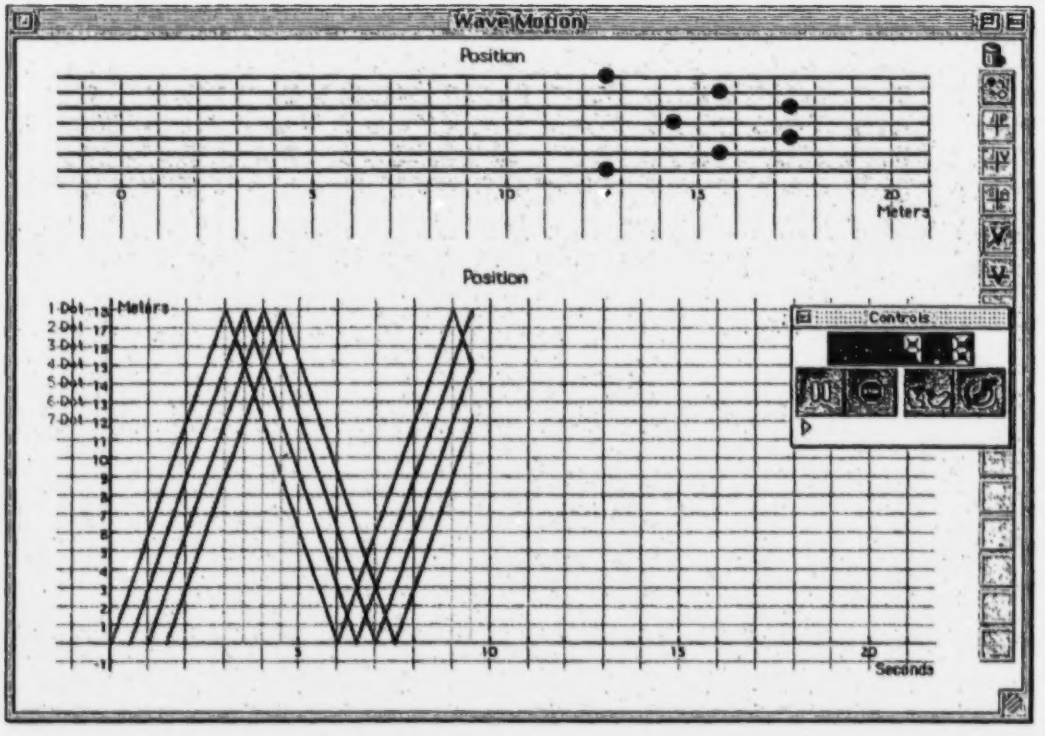

Figure 4. Wave—Staggered Starting Times

More Traditional Topics Using Participation in Shared Mathematical Objects. A standard student difficulty is in appreciating what it really means for a point to be on a line defined by an equation or other constraint. In the activity depicted in Figures 5 & 6, I "personalizes" this idea in a networked classroom where each student can send data to the teacher's computer (or calculator) as follows. First, students count-off in class, so each takes a number, which will serve as their x-coordinate. Then they are asked to make a point with their personal number as the x-coordinate but with whatever y-coordinate they wish, and send it up to the teacher's display. This results in the scattered points in Figure 5(a). Next, they are asked to make their y-coordinate *double their x-coordinate* and send this up. The result is the transformation depicted in Figure 5(b), where all the points assume their position on a line—which, of course, is the line we come to call "y = 2x". Naturally, any errors will show up as outliers.

Figure 5(a).

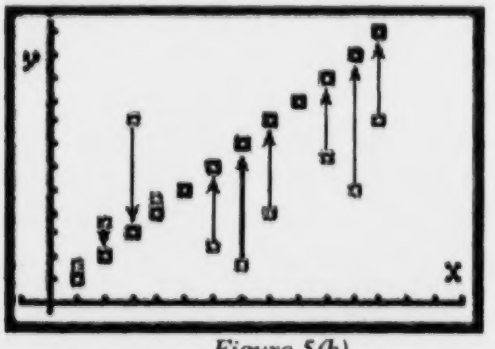

Figure 5(b).

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The sequence is repeated in Figure 6, where a new scatter of points appears in (a) and then the students are asked to make the second coordinate of their point to be the square of half their number. Here, the order of halving and squaring is important, and anyone who squares first and then takes half will not lie on the line. Indeed, the issue of order of operations turns into the issue of the identity of the appropriate line. Then, when the line is finally determined, we discover that one person, Damien, who's number is 8, is on both lines! Why? This becomes a lead-in to the matter of simultaneous equations, where Damien satisfies both constraints—the double of her number is the same as half of her number, which is then squared. The fact that this is true only for her and nobody else (adopting zero and negative numbers comes a bit later) is a reflection of the fact that she and only she satisfies the equation $x/2 = (x/2)^2$. We can then ask, what could we do to the rule so that Jeri is on both lines? And how much algebra can be done in this kind of connected classroom?

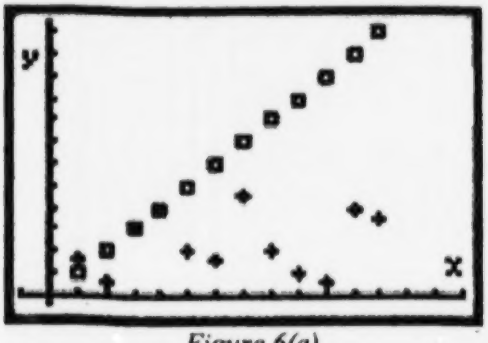

Figure 6(a).

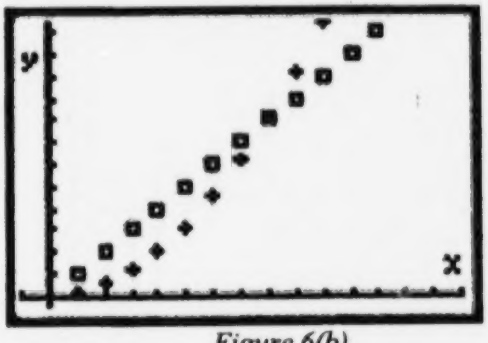

Figure 6(b).

We are currently examining, along with others, e.g., Stroup and Wilensky, how best to exploit the affordances of networked classrooms. The Participatory Simulations Project led by Stroup and Wilensky has been pursuing the opportunities for modeling emergent phenomena in dynamical systems, especially agent-based systems. For example, students can play the roles of predator or prey in a dynamic population model, or players in an economic model, and so on (Wilensky & Stroup, 2000). This work as well as that of another participatory simulations project based at the MIT Media Lab led by Resnick and Colella exploits parallel processing software that enables each participant to be represented as an independently controlled agent in the system.

Reflections on the Examples—Network-Based Activity Frameworks that Integrate Social Structure and Mathematical Structure

The above illustrations are only a small sliver of the possible range of activity structures in a networked classroom. Below is a further elaboration of the possibilities, limited due to space considerations.

Student-Teacher Activity Structures Using the Teacher's Computer as Publicly Viewable Common Ground.

(A) Students create functions on their own device and publicly upload them to a shared, publicly displayed object on the teacher's computer for a variety of purposes, including:

- Students contribute to an emergent object, where the properties or identity of the object are not well understood beforehand, and where the determination of the

properties and identity is at the heart of the learning opportunity as in the last example involving emergent lines and intersections.

- Students contribute to an object of their own design where the design work is at the heart of the learning opportunity as in the parade and marching band examples.

- Students contribute to an object where the identity and properties of the object are known in advance and where they act as the scaffolding for learning of something else, as might occur when the parade design and motions are well established, but where one half the class is defining the motion of their actors via position functions and the other half is using velocity functions. In this case the target is well-defined, and the heart of the learning opportunity is in defining the means by which the target is reached.

- Students upload survey or other data (e.g., probability trials) to a common data-set on the teacher's computer that is then aggregated and downloaded as a data object subject to further analysis by the students on their local devices. This could take place inside the classroom or, depending on connectivity, engage students elsewhere. Indeed, this is a possibility in almost all the activity structures, although some would be more convenient if the students were close-at-hand.

(B) Target Activities between teacher and students where students upload responses to classroom challenges, where challenges and responses are shown on the teacher's display.

- Define a function to fit this data, or an equation for this curve, or a polynomial that has these roots, etc. How many of these Gray Globs can you hit with one quadratic function? (Dugdale, 1982). Define a velocity (or position or acceleration) function to match this motion, ...

(C) Miscellaneous teacher-directed activities that will utilize the teacher's existing repertoire of classroom moves, e.g., pool solutions to an open ended problem and investigate solutions for generalities, optimality, etc.

Student Target Activities Between Students or Small Groups of Students. Essentially all of the whole-classroom activities have student-student analogs, either between single students working in pairs or, more likely, between small groups of students.

We have only had a brief glimpse into the learning opportunity space opened up by classroom connectivity. Time will be needed for the technology to be tuned to the possibilities and to the realities of such classrooms. But even more challenging is the need to understand how the social structures of the classroom and the mathematical structures can be made to interact in fruitful ways. At this point, experience is very limited, and we inherit a long tradition of three modes of activity: (1) Teacher-centered classroom activity, (2) Small-group activity, and (3) Individual activity. In all three cases, the communication among participants is biased towards oral communication, typically of an indirect nature about the mathematical objects and relations being used or studied. The new connectivity obviously offers much more direct communication. We are currently pursuing research that includes several private sector partners to examine the affordances and constraints of networked mathematics classrooms employing mixes of hardware and software platforms.

Part IV: Looking Ahead—Research Questions and Agendas

Introduction: The Bigger Picture

After approximately a generation of growing computer use in the world of business, LAN and WAN connectivity *coupled with the integration of computation into all aspects of business practice* has paid off in surprising increases in economic productivity during the past decade, now approximately 4% annually in the U.S. And, of course, the connectivity embodied in the WWW has led to even more startling impacts on the world outside of education. Indeed, this wider connectivity has changed the conditions of innovation in ways that compound and accelerate change (Bollier, 2000). We are poised to begin a comparable application of connectivity in education. The missing ingredients are at-hand computation (see, for example, Becker, *et al.*, 2000) and connectivity at the epicenter of teaching and learning, the classroom. But of course, as was the case in business, and as our brief analysis of the printing press suggested, many changes must take place across many different dimensions before a new representation infrastructure delivery technology can have full impact. The classroom connectivity ingredient can pay off only if coupled with the integration of computation into educational practice.

The illustrations in Part III focused primarily on student learning, so we need also to address what connectivity among diverse inexpensive computing devices might mean, not only for learning, but for teaching, classroom management and assessment—the broader ingredients of teaching practice. Each is a complex matter, and we cannot delve into detail here, but will simply offer some major issues needing investigation.

Research Issues and Opportunities in Assessment, Learning & Teaching: Three Opportunity Spaces

We need to understand how new configurations and applications of connected devices can support or perhaps impede potentially profound progress in three opportunity spaces at the communicative heart of mathematics education in real classrooms:

Diagnostic assessment and evaluation: We need to study how teachers can use connectivity and analytic tools to exploit what we know about student thinking and learning in order to actively diagnose and efficiently respond to student thinking on a regular basis in the classroom.

Student learning and activity structures: We need to study classroom affordances and constraints of new activity structures that exploit the ability of students to design and pass structured mathematical objects (e.g. functions) and representations (e.g. graphs) among themselves and to the teacher—as illustrated briefly above.

Teaching and the classroom management of information flow: We need to study the classroom management implications of wirelessly connected hand-helds, with particular attention to teacher-specific tools to help organize the flow of the vast amount of information available to them (e.g. what each student is doing on their individual hand-held), decide among alternative actions (e.g. send every student an identical graph vs. call students' attention to a projected graph on a central display), and set policies on

network communication (e.g., Can students send each other text messages? Only within their group? Only to an assigned partner?)

Specific Kinds of Concrete Research Questions Needing Investigation

Across these opportunity spaces, we need to gather and analyze data addressing the following broad kinds of questions in ways that span technology-specifics.

- What uses of mathematical notations and representations, when shared across devices, lead to deep, intense or efficient content-oriented interaction and meaning-making among students and between teacher and students?
- Which characteristics of networked, hand-held devices (e.g. screen size, lack of color, availability of stylus, ease of beaming data, ability to move about the room) strongly enable or impede the ease, comfort, and effectiveness of mathematical conversations in the classroom?
- In what ways do networked, hand-held devices most strongly engage learners' cognitive strengths and solve practical, important teaching problems, or conversely, distract learners from the task at hand and impose new burdens on the teacher?

Importantly, given the novelty of these environments and technologies, we cannot trap ourselves into study of phenomena that will disappear when the technologies become better established. Hence we need to analyze how the answers to the above questions change as both the teachers' experience with connectivity, and the technologies themselves, mature. These research questions reflect the belief that hand-held, networked devices will not necessarily have a simple causal effect upon learning outcomes. As has been the case historically, introducing technology into schools will not necessarily change practice (Cuban, 1986), and it is likewise well appreciated that many technological tools simply reinforce existing practice (Marx, *et al.*, 1998; Means, 1994). Similarly, as argued by Roschelle & Pea (1999) regarding WWW connectivity with resources outside the classroom, the promise may be ill-understood or near-sighted.

Hence, we need to begin building a framework that can adequately describe *uses* that are likely to distinguish effective from ineffective practice. Likewise, we cannot take for granted that manufacturers specifications (processor speed, communication speed, screen size) are the device characteristics that necessarily enable or impede use, and should seek to build a *conceptual analysis of device characteristics* that directly relates to observable classroom behaviors.

Finally, widespread impact from such devices is only likely if we identify the strongest ties to learner's strengths, solve difficult teaching problems, and introduce no serious *new* difficulties. Thus we need conceptual and analytic frameworks to clearly identify how these affordances and constraints play out in realistic classroom settings, and thereby to guide iterative design that magnifies the unique benefits and minimizes the newly introduced impediments. This will in turn inform the design of appropriate teacher development and support structures so that connectivity becomes a pedagogical support tool. As we begin to understand the classroom issues and technological issues we must immediately employ these understandings in the development of teacher development and support systems.

Last Words: Recognizing the Depth of the Changes that Are Underway

We are early in an exciting new era for technology in mathematics education. Both the representational infrastructures are changing and the physical means for implementing them are changing. We are seeing new alphabets emerging, new visual modalities of human experience are being engaged, and new physical devices are emerging—all at the same time. Much work needs to be done.

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